Pergamon

Topology Vol. 33, No. 2, pp. 257–270, 1994

0040-9383/94.$6.00 + 0.00

KNOTS, BRAID INDEX AND DYNAMICAL TYPE

JÉRÔME E. LOS

(*Received* 20 *December* 1991; *in revised form* 20 *June* 1993)

1. INTRODUCTION

THE presentation of links in S^3 by closed braids has been widely studied over the years and yet some new developments still appear, for instance, the recent series of papers by Birman and Menasco ending with [5, 6].

In the present paper we consider another aspect of the closed braid presentations, namely the dynamical properties of the braids representing a given knot and with the minimal number of strands called the *braid index*. This set of braids is called the *braid index set* and is denoted by $Bi(K)$.

The relationship between braids and dynamics has been studied since Artin [1], in the sense that a braid β ($\beta \in B_n$) defines a unique isotopy class of homeomorphisms $[f_\beta]$ on the n-punctured disc. On the other hand, isotopy classes of surface homeomorphisms are classified by the Nielsen–Thurston theorem [21, 23] into three *dynamical types:periodic*, *pseudo-Anosov* and *reducible*. This theorem induces a natural classification for braids. The word reducible in the context of the braid groups could be confusing. In what follows the word reducible will only be used in the dynamical sense.

It is tempting to seek a related classification for knots by the dynamical type of their minimal braid representatives. The restriction to the braid index set is natural although not obvious in practice. Indeed, without the restriction to $Bi(K)$ one can find a braid representative in each of the above dynamical types for the trivial knot (Fig. 1).

Knots can be classified in many ways. For example the geometry of the knot complement and therefore, knots, is classified by a theorem of Thurston [24] which is very similar to the above one, i.e. there are three classes: the torus knots, the satellite knots and the others. The result says that the knot complement of "the others" has a hyperbolic structure. These knots will be called *hyperbolic knots*.

In what follows, a special class of satellite knots will appear. This class is well known and is related to the type 0 embedding in the Birman–Menasco terminology [6] (see §3). If K is a satellite knot, $S^3 - K$ admits a collection $\{T_1, \ldots, T_k\}$ of essential non peripheral tori, which are nested since K is a knot. This means that each T_i bounds a solid torus $\mathbf{T}_i$ and $\mathbf{T}_1 \supset \ldots \supset \mathbf{T}_k$.

The knot K is said to be a *braided satellite knot* if for some $j \in \{2, \ldots, k\}$, T_j intersects transversally each fiber, $D^2_{j-1} \times \{\theta\}$, of the solid torus $\mathbf{T}_{j-1} = D^2_{j-1} \times S^1$, along a meridian. This torus T_j is said to be *braided*, indeed its core is a closed braid in the solid torus $\mathbf{T}_{j-1}$. In order to state our result we now define pictorially a transformation called an *exchange move* by Birman–Menasco. This transformation is central in their series of papers [5, 6] as well as here (see Fig. 2). A braid $\beta \in B_n$, as a word in the generators $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$ of B_n is said to *admit an exchange move* if β is conjugate to $\gamma = \sigma_{n-1}^{\pm 1} \cdot P \cdot \sigma_{n-1}^{\mp 1} \cdot Q$, where P and Q are words in $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-2}^{\pm 1}\}$.

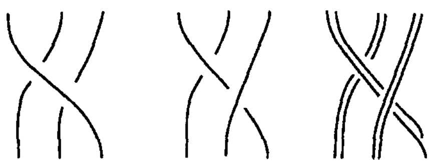

Fig. 1. A periodic (a), a pseudo-Anosov (b) and a reducible (c) braid which represents the trivial knot.

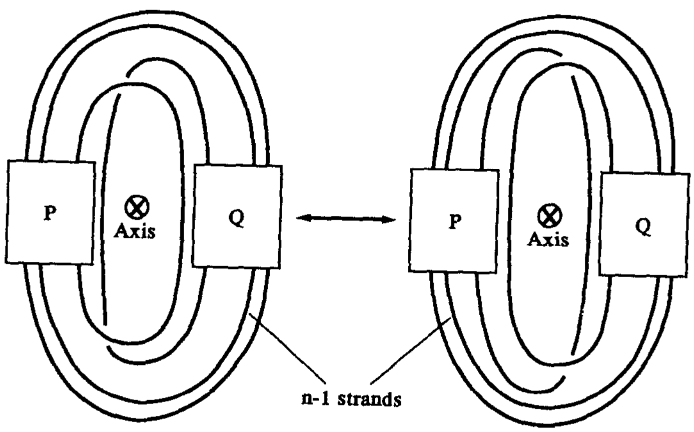

Fig. 2. An exchange move.

THEOREM 1.1. *Let K be a knot in S^3 and let $Bi(K)$ be its braid index set, then one of the following occurs*:

(i) *K is a torus knot if and only if every braid in $Bi(K)$ is periodic.*
(ii) *K is an hyperbolic knot or a non braided satellite knot if and only if every braid in $Bi(K)$ is pseudo-Anosov.*
(iii) *K is a braided satellite knot if and only if for every $\beta \in Bi(K)$ there exists a finite sequence of exchange moves (and conjugacies) in $Bi(K)$: $\beta \to \beta_1 \to \ldots \to \beta_N$ such that β_N is reducible.*

One could expect a stronger statement in case (iii) but we shall see, during the proof (§4), that the exchange moves are generally unavoidable. In this paper we consider only knots rather than the more general case of links because of orientation problems, nevertheless the arguments work in the same way for links with suitable orientation. From Theorem 1.1 one can easily derive the following:

COROLLARY 1.2. *If K is a torus knot then the braid index set $Bi(K)$ is reduced to a single conjugacy class.*

From torus knots one can construct a large class of braided satellite knots whose braid index set is reduced to a single conjugacy class, for example all the iterated torus knots. Theorem 1.1 enables one to define a non trival knot invariant from the dynamics of the braids in $Bi(K)$. Indeed, for any braid $\beta \in B_n$ let $[f_\beta]$ be the corresponding isotopy class of homeomorphisms of the n-punctured disc. The number

$$\lambda_\beta = \inf\{\exp(h(\varphi)) \colon \varphi \in [f_\beta]\}, \tag{1}$$

where $h(\cdot)$ is the topological entropy of the map (see [2]), is called the *dilatation factor of the braid.* This is a conjugacy class invariant which has been studied in [16] for any surface homeomorphism. In that case one defined a link invariant of a class of links, called braided links which generalize fibered links in S^3 (see also [22]). In this paper λ_β is used to define the following invariant for any knot (or link) K in S^3:

$$\Lambda_K = \inf\{\lambda_\beta : \beta \in Bi(K)\}. \tag{2}$$

Another consequence of Theorem 1.1 is:

THEOREM 1.3. *Let K be a knot in S^3, then:*

(i) $\Lambda_K = 1$ *if and only if K is a torus knot or an iterated torus knot.*
(ii) *If K has braid index n and is not a torus knot or an iterated torus knot then* $\Lambda_K \geq 2^{1/n}$.

The dilatation factor of a given braid is a computable number by using recent algorithms [3] or [17]. Unfortunately this is not the case for the braid index and, *a fortiori*, for the braid index set. Nevertheless there are some cases where the factor Λ is computable. In the §5 we prove that the granny and the square knots have different Λ-factor and we compute them. In fact the factor Λ is computable for all braid index 3 links by using the Birman–Menasco classification theorem [7] for braid index 3 links.

2. DYNAMICS AND BRAIDS

A brief review

Let us begin this section by recalling the Nielsen–Thurston classification theorem [21, 23] (see [11]) for orientation preserving surface homeomorphisms $f: S \to S$. Let $[f]$ denote the isotopy class of f. A homeomorphism f on a compact surface S is said to be *reducible* if it leaves invariant a non empty collection C of disjoint essential simple closed curves of S, called a *reducing manifold.* If f is reducible then each component S_i of $S - C$ is invariant under a least iterate f^{k_i}. The isotopy class of f^{k_i}, restricted to the component S_i (actually, it's compactification) is called a *component* of $[f]$. An isotopy class is *irreducible* if it contains only one component. The Nielsen–Thurston theorem can be stated as follows:

THEOREM 2.1. *Let S be a compact orientable surface and $[f]$ an isotopy class of orientation preserving homeomorphisms of S then one of the following situations occurs:*

(i) *$[f]$ is periodic: for a positive integer m, f^m is isotopic to id_S.*
(ii) *$[f]$ is pseudo-Anosov: f is isotopic to a pseudo-Anosov homeomorphism.*
(iii) *$[f]$ is reducible: there exists a reducing manifold C for some $\varphi \in [f]$ such that every component is irreducible, i.e. they satisfy either* (i) *or* (ii).

The precise properties of the pseudo-Anosov maps are not explicitly used in this paper. Nevertheless, for convenience, we recall the definition. A measured foliation (F, μ) on a surface S, is a foliation F with a finite set of singularities together which a transverse measure μ, i.e. a measure defined on each arc transverse to the leaves of F. *A* homeomorphism f on S is *pseudo-Anosov* if there exists a pair of transverse measured foliation (F^s, μ^s), (F^u, μ^u) and a real number $\lambda > 1$ such that:

$$f(F^s, \mu^s) = (F^s, \lambda^{-1} \cdot \mu^s) \text{ and } f(F^u, \mu^u) = (F^u, \lambda \cdot \mu^u).$$

The measured foliation (F^s, μ^s) (resp (F^u, μ^u)) is called the *stable* (resp. *unstable*) *invariant foliation* and the number λ is called the *dilatation factor* of the pseudo-Anosov homeomorphism.

Pseudo-Anosov maps possess many beautiful properties. The one which will be used here has been found by Fathi and Shub in [11 exposé 10] and concerns the topological entropy. The topological entropy is a conjugacy invariant for continuous maps on compact manifolds. The orginal definition can be found in [2] (see [8] for an equivalent definition). The topological entropy $h(f)$ of a map f is a non-negative number which measures the "asymptotic complexity" of the dynamics. In particular, for maps on the interval or homeomorphisms on surfaces, positive topological entropy is detected by the existence of infinitely many periodic orbits whose number grows exponentially with the least period (see [15]). This is not true in higher dimension. The basic properties of the topological entropy are:

(3a) for every positive integer, $h(f^n) = n.h(f)$, and for an homeomorphism $h(f^{-1}) = h(f)$,
(3b) if $f\colon X \to X$ has an invariant submanifold Z, then $h(f) = \max\{h(f_{/Z}), h(f_{/X-Z})\}$.

For surface homeomorphisms the topological entropy enables one to define the following isotopy class invariant:

$$\lambda[f, S] = \inf\{\exp(h(\varphi) : \varphi \in [f]\}, \tag{4}$$

where $h(\cdot)$ is the topological entropy (S may be non connected). This invariant is called the *dilatation factor* and the following properties are easy to derive from the properties (3) above and the results of [11, exposé 10]:

(5i) If $[f]$ is periodic then $\lambda[f, S] = 1$.
(5ii) If $[f]$ is pseudo-Anosov then $\lambda[f, S]$ is the dilatation factor of the pseudo-Anosov representative of the class, and then $\lambda[f, S] > 1$.
(5iii) If $[f]$ is reducible, let $[f^{k_i}{}_{/S_i}]$, $i = 1, \ldots, N$ be the irreducible components (all k_i are minimal) then $\lambda[f, S] = \max\{(\lambda[f^{k_1}{}_{/S_1}, S_1])^{1/k_1}, \ldots, (\lambda[f^{k_N}{}_{/S_N}, S_N])^{1/k_N}\}$.

Let us now recall the well known relationship between braids and dynamics (see [4] for the details). Assume that the surface S above is the n-punctured disc D_n, i.e. the complement of n open discs in D^2. We denote by ∂D the outer boundary. Let $\mathrm{Aut}(D_n)$ be the group of orientation preserving homeomorphisms of D_n, fixing ∂D setwise and let A_n be the set of path components of $\mathrm{Aut}(D_n)$ (isotopy classes). The mapping class group $M(0, n)$, as defined in [4], is the subgroup of A_n whose elements fix ∂D pointwise. The kernel of the inclusion map $M(0, n) \to A_n$ is generated by a Dehn twist in a neighborhood of ∂D.

There are many ways to define the n-strand braid group B_n. The most economic is the algebraic presentation:

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle.$$

The homomorphism $B_n \to A_n$ obtained by mapping each generator $\beta = \sigma_i$ to $[f_\beta]$, the Dehn twist that exchanges the ith puncture and the $(i+1)$st puncture is an epimorphism. Its kernel is generated by the "full twist" $\Delta^2 = (\sigma_1\sigma_2 \ldots \sigma_{n-1})^n$ (see [4] 1.10). Using this epimorphism we pull back the Nielsen–Thurston classification of A_n to a classification of elements of B_n, so that it is meaningful to say that a braid is periodic, reducible or pseudo-Anosov.

Proof of Theorem 1.3. The factor Λ_K defined by (2) in the introduction is a knot invariant by definition. We shall prove that Theorem 1.1 implies it is a non trivial one. The estimate given by Theorem 1.3(ii) is a consequence of the following result:

THEOREM 2.2. [18]. *If f is a pseudo-Anosov homeomorphism of the n-punctured disc D_n (then $n > 2$) then its topological entropy satisfies*: $h(f) \geq \frac{1}{n} \log 2$.

Let us prove part (i). Assume that $\Lambda_K = 1$ then there is a braid β in $Bi(K)$ such that $\lambda_\beta = 1$. Indeed $\lambda = 1$ cannot be an accumulation point by 2.2. Therefore β is either a periodic braid or a reducible braid but with all components periodic by (5ii). In the first case $K = \bar{\beta}$ is a torus knot by 1.1(i) and in the second case K is an iterated torus knot. The converse is also very simple. In this case, by Corollary 1.2, there is only one conjugacy class in the braid index set. For torus knots the result is a consequence of 1.1(i) and property (5i). For iterated torus knots, one braid of $Bi(K)$ is obtained by a sequence of "cablings" of periodic braids (see §5 for details). The resulting braid is reducible and each component is periodic. The result follows from (5i) and (5iii).

For part (ii), there are also two cases:

(a) K is a hyperbolic or a non braided satellite knot in which case all the braids in $Bi(K)$ are pseudo-Anosov by 1.1(ii). Theorem 2.2 applies directly.

(b) K is a braided satellite knot. By 1.1(iii), the braids in $Bi(K)$ are either pseudo-Anosov or reducible. In the first case the result follows from 2.2. In the second case there is at least one irreducible component which is pseudo-Anosov. Assume, for simplicity, that the isotopy class $[f_\beta]$ of the reducible braid has only two irreducible components. The reducing manifold is a collection of essential curves $\{C_1, \ldots, C_k\}$ which are permuted in a single cycle under one element Ψ of $[f_\beta]$. Each C_i bounds a punctured disc D^i with $r = n/k$ punctures. The two irreducible components are now easy to describe:

$$(*) \quad [f_{\beta,1}] = [\Psi_{/[D^2 - \cup \mathring{D}_i]}], \qquad \text{and} \qquad (**) \quad [f_{\beta,2}] = [\Psi_{/[\cup D_i]}].$$

If $[f_{\beta,1}]$ is a pseudo-Anosov component then by 2.2 and ([11 exposé 10):

$$h(\varphi) \geq \frac{1}{k} \log 2 > \frac{1}{n} \log 2,$$

where $h(\varphi)$ is the topological entropy of any element of the isotopy class $[f_{\beta,1}]$. The result follows in this case from property (3(b)) of the topological entropy.

If $[f_{\beta,2}]$ is a pseudo-Anosov component, which means that the classes $[\Psi^k/D_i]$ are pseudo-Anosov (for any i), then by 2.2: $h(\varphi) \geq 1/r \log 2$ ($r = n/k$), for any element of the isotopy class $[\Psi^k/D_i]$. The result follows, in this case, from the properties (3a) and (3b) of the topological entropy.

In the general case with any number of irreducible components, the above argument is applied to each component.

3. BRAIDS AND LINKS

In this section we recall some known results about braids and links which will be used in the proof of Theorem 1.1. The first one is due to Morton [19] and Franks and Williams [12]. It relates the braid index to the degree of the two variables Jones polynomial $X_L(q, \lambda)$ (see [14] or [13]).

THEOREM 3.1. [19, 12]. *Let L be an oriented link with polynomial $X_L(q, \lambda)$. If d_+ and d_- are, respectively, the largest and the smallest degree of λ in X_L and if L is the closure of a braid $\alpha \in B_n$ then:*

$$n \geq d_+ - d_- + 1.$$

The second result we will use is the description, given by Birman and Menasco [6], of the possible embeddings of essential tori in the complement of a link L, presented as a closed braid. We consider a link L as a closed n-braid with braid axis A. We denote by D_θ the fibers of the solid torus $\mathbf{T}_A$ which is the complement of the axis A.

Type 0 *tori.* Let T be an essential torus bounding a solid torus $\mathbf{T}$ whose core is a closed braid with axis A. Then T is transverse to every fiber of $\mathbf{T}_A$ and intersects each D_θ in a meridian.

Type 1 *tori.* Choose a 2-sphere S which is pierced twice by the axis A and with a standard south-north foliation induced by $D_\theta \cap S$. Consider an arc α in the solid torus $\mathbf{T}_A$ which is transverse to the fibers D_θ and connects S to itself. The torus T is obtained by tubing S to itself along a neighborhood of α.

Type k tori ($k \geq 2$). The torus T is made up of 2-spheres $S_1, \ldots, S_k$ cyclically connected by cylinders $C_1, \ldots, C_k$ such that:

(i) The spheres $S_1, \ldots, S_k$ are disjointly embedded in S^3 so that each S_i intersects the axis A twice and has a standard north–south foliation induced by $D_\theta \cap S_i$.
(ii) The cylinders $C_1, \ldots, C_k$ are tubular neighborhoods of arcs $\alpha_1, \ldots, \alpha_k$ which connect the S_i's in a cycle such that each α_i lies in a fiber D_{θ_i}.

THEOREM 3.2. [6] *Let L be a link in S^3 which is the closure of an n-braid β and assume that $S^3 - L$ has a collection $\{T_1, \ldots, T_r\}$ of essential tori. Then there exists a finite sequence of braids in B_n: $\beta \to \beta_1 \to \ldots \to \beta_m$ such that:*

(i) *Each β_{i+1} is an n-braid with axis A_{i+1} and is obtained from β_i by an exchange move and a conjugacy.*
(ii) *Each essential torus T_j is isotopic, in the complement of $\bar{\beta}_m \cup A_m$, to a torus of type* 0, *type* 1 *or type k.*

It is of course not the place here to recall the proof of this theorem, nevertheless, let us say some words about the idea of the proof. Consider one torus T_i of the above collection and its embedding in the complement of the closed braid $\bar{\beta}$. The axis A of β intersects T_i at some isolated points and we can assume that the intersections are transverse. Each fiber D_θ of the braid fibration intersects T_i. These intersections are transverse except for finitely many singular points and $D_\theta \cap T_i$, $\theta \in S^1$ defines a singular foliation of the torus. The proof of the theorem is an analysis of these singular foliations. The first step is to show that any singular foliation can be transformed into a "standard" one by a finite sequence of elementary exchange moves and conjugacies. The second step is to prove that there are only three types of standard foliations. Finally the last step is to relate the standard foliations of the torus with the way the torus is embedded in 3-space.

4. PROOF OF THE MAIN THEOREM

In addition to the braid index set $Bi(K)$, let us also consider the set $M(K)$ of all the braids in $\cup B_n$ whose closure is the knot K.

LEMMA 4.1. *K is a torus link if and only if there exists a braid β in $M(K)$ which is periodic.*

If $\beta \in B_n$ is periodic then there is an element φ in the isotopy class $[f_\beta]$ defined by β which is a rigid rotation of the disc [9]. The punctured disc D_n admits a foliation by circles which is invariant under φ. The punctures $\{z_1, \dots, z_n\}$, which we consider as isolated points, belong to one of these circles C. Therefore, the n strands of β can be isotoped to lie on the cylinder $C \times [0, 1]$. Closing the braid defines a link on the torus $C \times S^1$.

Conversely if K is a torus link it lies on a standardly embedded torus T. A meridian of T defines an axis A and K is a closed braid $K = \bar{\beta}$ relative to A. The torus T intersects each fiber D_θ of the solid torus $\mathbf{T}_A = S^3 - A$ in a circle C_θ containing, for all θ, the points of $K \cap D_\theta$. Therefore C_0 is invariant under some homeomorphism φ in the isotopy class $[f_\beta]$ defined by the braid β. Furthermore, $\varphi_{/C_0}$ is a rotation permuting the points of $K \cap D_0$. The components of $D_n - C_0$, where $D_n = D_0 - (K \cap D_0)$, are a disc and an annulus. By an isotopy from φ to φ' we can extend the rotation on C_0 to a rigid rotation on the disc D_n.

LEMMA 4.2. *If K is a torus knot then there exists a braid β in $Bi(K)$ which is periodic.*

Assume that K is a (p,q) torus knot with $(p,q) = 1$ and $q < p$. The standard braid representative of K: $\beta_{p,q} = (\sigma_1 \sigma_2, \dots, \sigma_{q-1})^p \in B_q$, is periodic. It remains to prove that q is the braid index. To this end the inequality (3.1) of Morton–Franks–Williams is sufficient. Indeed in [14] Jones has computed the polynomial X_L for any (p,q) torus knot and from this formula we have $d_+ - d_- = q - 1$ proving the Lemma.

LEMMA 4.3. *If there is a reducible braid in the braid index set $Bi(K)$ then the knot K is a braided satellite knot.*

If $\beta \in Bi(K)$ is reducible then there is a collection $\{C_1, \dots, C_k\}$ of essential simple closed curves in the disc D_n which is setwise invariant under an element of the isotopy class $[f_\beta]$. Closing the braid defines a collection of tori which are obviously embedded as the type 0 tori. It remains to show that these tori are essential. Each of these tori bounds a solid torus whose intersections with the discs D_θ are punctured discs with more than one puncture since the curves C_i are essential in the punctured disc. The tori are then not boundary parallel. By minimality of the number of strands these tori are knotted, proving the Lemma.

From now on we shall use the Birman–Menasco theorem (3.2) as an essential tool.

PROPOSITION 4.4. *If K is a braided satellite knot then for any braids β in $Bi(K)$ there exists a finite sequence of exchange moves and conjugacies: $\beta \to \beta_1 \to \dots \to \beta_N$ such that β_N is reducible.*

In the terminology of [6] this result means that the types 1 and k embeddings cannot occur for braided satellite knots presented as closed braids with the minimal number of strands.

The proof of Theorem 3.2 is local in the sense that each torus of the decomposition is studied independently of the others. We can then reduce the study to the case where there is only one essential torus T. In this case T bounds a solid torus $\mathbf{T}$ whose fibers, which we call δ_φ, are transverse to K.

We have to study the embedding of the knot K with respect to two distinct fibrations:

(a) The braid fibration, i.e. the fibration of the axis complement whose fibers are called D_θ.
(b) The fibration of the solid torus $\mathbf{T}$ whose fibers are denoted δ_φ.

Since K is a closed braid with axis A then K is transverse to all the fibers D_θ. Furthermore K is a braided satellite knot then, after an isotopy in the solid torus $\mathbf{T}$, K is also transverse to all the fibers δ_φ.

Assume first that T is a torus of type k, eventually after a finite sequence of exchange moves and conjugacies. By definition there are k disjointly embedded 2-spheres $S_1, \dots, S_k$, which are connected by cylinders $C_1, \dots, C_k$. Each C_i being a tubular neighborhood of an arc α_i embedded in a fiber D_{θ_i}. Then the cyclinder C_i intersects the fibers D_θ of the braid fibration for $\theta \in [\theta_i - \varepsilon, \theta_i + \varepsilon]$, where $\varepsilon > 0$ is small. Since K is a braided satellite knot, for a given orientation of K all the intersections $K \cap \delta_\varphi$ have the same sign in each cylinder C_i. Each C_i encloses more than two strands since the torus T is not boundary parallel. Moreover K is a closed braid, then all the intersections $K \cap D_\theta$ also have the same sign. Therefore the knot K is embedded in each cylinder C_i as shown by the Fig. 3(a).

A given sphere S_1 is connected to the other spheres by two cylinders let say C_1 and C_2. Assume now that the strands which are enclosed into C_1 leave C_1 and arrive in the ball B_1 bounded by S_1 (with an obvious abuse of terminology).

Since K is a closed curve then all the strands arriving into B_1 from C_1 have to leave B_1 either by (α) entering into C_1 or (β) by entering into C_2.

Claim 1. The case (α): $C_1 \to B_1 \to C_1$ is not possible (see Fig. 3(bi).

Indeed from the local embedding of K given by (i) there is a fiber δ_φ of the solid torus $\mathbf{T}$, which is contained into the ball B_1 such that K is tangent to δ_φ. Since K is a braided satellite knot this tangency can be removed by an isotopy of the solid torus. Under this isotopy, the backtracking strand has to cross the axis A and then can be removed from the cylinder C_1. This is not possible since otherwise the number of strands of the braid would decrease.

From the claim 1 all the strands arriving in the ball B_1 from C_1 have to leave B_1 by the cylinder C_2. Three situations can occur:

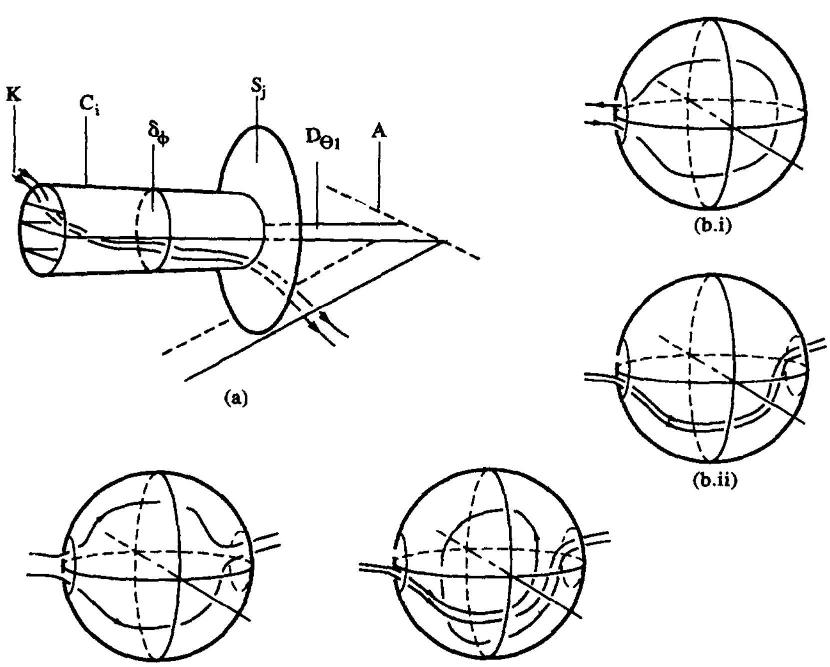

Fig. 3. The torus cannot be an embedding of type k.

(1) All the strands from C_1 to C_2 in the ball B_1 crosses the same fibers D_θ of the braid fibration, $\theta \in [\theta_1 + \varepsilon, \theta_2 - \zeta]$ for some $\varepsilon, \zeta > 0$ (see the Fig. 3(bii).
(2) Some of the strands from C_1 to C_2 in the ball B_1 crosses the fibers D_θ in a positive direction and some others crosses the fibers D_θ in a negative direction, i.e. either for increasing θ or for decreasing θ (see Fig. 3(biii).
(3) Some of the strands from C_1 to C_2 in the ball B_1 turn around the axis more than once, i.e they cross the fibers D_θ for $\theta \in [\theta_1 + \varepsilon, \theta_2 - \zeta + 2k\pi]$ for some $\varepsilon, \zeta > 0$ and $k \geq 1$ (see the Fig 3(b(iv).

Claim 2. The cases (2) and (3) above are not possible.

The case (2) is not possible since otherwise K would not be a closed braid. The case (3) is not possible by an agrument similar to the proof of the Claim 1. Indeed for this local embedding there are also tangencies between K and some fibers δ_φ of the solid torus $\mathbf{T}$. Each tangency can be removed by an isotopy in the torus T since K is a braided satellite knot. This isotopy is actually a Markov move decreasing the number of strands.

Therefore, the only possible embedding in each ball B_i is given by the case (1) above. In this case all the intersections $A \cap S_i$ can be removed by an isotopy and the intersection $T \cap D_\theta$ for each θ is a collection of closed curve in the disc. This implies that T is a torus of type 0.

By exactly the same arguments we prove that the torus T cannot be of type 1, which completes the proof of the proposition.

Notice that the assumption that K is a knot has been used here because some other cases may arise with links, depending upon the orientations of the components.

The parts (ii) and (iii) of Theorem 1.1 are now proved. Indeed 4.3 and 4.4 implies (iii). Furthermore from 4.1–4.4, a knot K which is neither a torus knot nor a braided satellite knot cannot have any periodic or reducible braid in its braid index set. Then, in this case, all the braids in $Bi(K)$ are pseudo-Anosov. The converse follows by the same lines.

It remains to prove the part (i) of the Theorem. In fact the Birman–Menasco theorem (3.2) is again the main tool. Indeed if K is a (p, q)-torus knot there is an embedded torus, namely the one on which K is embedded. This torus is trivially embedded in S^3, i.e. it bounds two solid tori $\mathbf{T}_p$ and $\mathbf{T}_q$. The torus T is oriented and then its normal bundle is well defined. Pushing the knot K along one normal direction gives a torus T_p or T_q embedded in $S^3 - K$. The Birman–Menasco analysis can be applied to this tori as well as for essential tori in the complement of a satellite knot. Indeed, as we noticed above, the analysis leading to the Theorem 3.2 deals with singular foliations on a torus induced by the braid fibration. The fact that a torus, in the complement of a knot, is knotted or not plays no role in the proof of Theorem 3.2. Therefore the proof of the Proposition 4.4 can be used directly for the torus T_p (resp T_q) above. To this end we just have to check that the knot K is braided with respect to the torus T_p (resp T_q) which is obvious. The difference between T_p and T_q is that T_p encloses p strands and T_q encloses q strands. Since K is a torus knot then $p \neq q$ (for instance $q < p$). The proof of the Proposition 4.4 can then be applied to the torus T_q, it gives:

Proposition 4.5. *If K is a torus knot then for all braids β in $Bi(K)$ there exists a finite sequence of exchange moves (and conjugacies): $\beta \to \beta_1 \ldots \to \beta_N$ such that β_N is periodic.*

The conclusion of the proposition means that the braid β_N is precisely the standard braid (up to conjugacy) we are expecting i.e. $(\sigma_1 \sigma_2 \ldots \sigma_{q-1})^p \in B_q$. There exists, in fact, only one such periodic braid in B_q since the rotation $R_{p/q}$ of D_q is uniquely defined by p and q. Theorem 1.1 and the Corollary 1.2 will be proved after the following:

LEMMA 4.6. *The braid $\beta_{p/q} = (\sigma_1\sigma_2 \dots \sigma_{q-1})^p \in B_q$, $p > q$, does not admit an exchange move.*

If $\beta_{p/q}$ admits an exchange move then it is conjugate to $\beta = \sigma_{q-1}^{\pm 1} \cdot P \cdot \sigma_{q-1}^{\mp 1} \cdot Q$, where P and Q are two braid words written with the letters $\{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \dots, \sigma_{q-2}^{\pm 1}\}$. Let us consider the writting of the braids $\beta_{p/q}$ and β in a more convenient way for our purpose (see [10]). Consider each strand of a geometric braid, numbered $\{1, \dots, q\}$ from left to right at the top level (i.e. on the disc $D^2 \times \{0\}$). The natural orientation of the braid (from top to bottom) defines an ordering of the crossings $\{1, \dots, N\}$. If the rth crossing involves the strands i and j then denote this crossing by the *strand letter* $s_{i,j}(r)^{+1}$ if i crosses over j from the left to right and $s_{i,j}(r)^{-1}$ if i crosses over j from right to left. A *strand word* for a braid β is the word $s(\beta) = s(1).s(2) \dots s(N)$, where $s(r)$ is the strand letter of the rth crossing. Notice that the strand letters are not generator of the braid group as are the σ_i. With this notation the braid $\beta_{p/q}$ can be written as:

$$s(\beta_{p/q}) = (s_{1,2} \cdot s_{1,3} \dots s_{1,q})\,(s_{2,3} \cdot s_{2,4} \dots s_{2,q} s_{2,1}) \dots\dots (p \text{ such blocs}).$$

There are three obvious relations between the $s_{i,j}$:

(i) $s_{i,j} \cdot s_{i,j}^{-1} = s_{j,i}^{-1} \cdot s_{j,i} =$ the empty word,
(ii) if $\{i,j\} \cap \{a,b\} = \varnothing$ then $s_{i,j}^{\pm 1} \cdot s_{a,b}^{\pm 1} = s_{a,b}^{\pm 1} \cdot s_{i,j}^{\pm 1}$,
(iii) for any i,j,k: $s_{i,j} \cdot s_{i,k} s_{j,k} = s_{j,k} \cdot s_{i,k} s_{i,j}$.

This writting has the advantage of keeping track of all the crossings of a given type (i,j). This is not the case for the equivalent relations with the generators σ_i. The proof of the Lemma is now easy. The braid $\beta_{p/q}$ is positive then it has the minimal number of crossings (length of the braid word) in its conjugacy class. Furthermore, if $p > q$ and $(p,q) = 1$ then $p = qm + r, m \geq 1$ and $0 < r < q$, and each letter $s_{i,j}$ in the strand word $s(\beta_{p/q})$ appears at least $2m$ times. Since $s(\beta_{p/q})$ has the minimal number of crossings in its conjugacy class, this inequality is satisfied by any braid in the conjugacy class. On the other hand the writing of the braid β above implies that for some $k \in \{1, 2, \dots, q-2\}$ the crossing $s_{k,q-1}$ appears only once, therefore β and $\beta_{p/q}$ cannot be conjugate. The Lemma is then proved as well as the theorem.

5. SOME PROPERTIES AND APPLICATIONS

Let us give some precisions on the main theorem. Consider a braided satellite knot K and assume, for simplicity, that K has only one essential torus T. The core of T is a non trivial knot C which is a closed braid $\bar{\beta}$ and K is a closed braid $\bar{\alpha}$ in the solid torus $\mathbf{T}$ bounded by T. Let us denote the knot K by: $K = \bar{\beta} \clubsuit \bar{\alpha}$ with $\alpha \in B_n$ and $C = \bar{\beta}$ with $\beta \in B_m$, then we notice that:

(i) the knot K is uniquely determined by the knot C and the conjugacy class of α in B_n,
(ii) the braid index $bi(K) = m.n$, and $m = bi(C)$.

The claim (i) is a consequence of a well known result of Morton [20] and (ii) follows from [6].

In the introduction we mentioned that the exchange moves, in the Theorem 1.1, are unavoidable. Let us explain this claim. Consider a braid $\beta \in Bi(C)$ and the knot $K = \bar{\beta} \clubsuit \bar{\alpha}$ for any braid α in B_n. This knot K is the closure of a reducible braid $\gamma \in Bi(K)$ which we write $\gamma = \beta \clubsuit \alpha$ and which is obtained as follows:

(a) Consider the strands of β embedded in the solid cylinder $D^2 \times [0,1]$ and let $N(\beta)$ be a regular neighborhood. $N(\beta)$ is a collection of disjoint embedded solid cylinders: SC_i, $i = 1, \dots, m$.

(b) In each cylinder SC_i, $i = 1, \dots, m-1$, consider the embedding of the identity braid of B_n (n parallel strands) and, in the remaining cyclinder SC_m, consider the embedding of the strands of α.

The resulting braid γ is obviously reducible and belongs to $Bi(K)$. In fact every reducible braid whose closure has a single essential torus is conjugate to a braid of this form [16]. Now we observe that if the braid β admits an exchange move then the braid γ also admits some exchange moves. Unfortunately the braids obtained by an exchange move from a reducible braid are not all reducible. Moreover the class of knots such that some braids in the braid index set admits an exchange move is not empty (see [5] for some examples). This explains the claim.

In fact the converse of this property is also true, i.e. if $\gamma = \beta \clubsuit \alpha$ admits an exchange move then β admits an exchange move. Indeed, from the proof of Lemma 4.6 there is a crossing $s_{i,j}$ which appears only once in the braid γ. Assume that $K = \bar{\gamma}$ is a knot and that β belongs to the braid index set $Bi(C)$ of a knot C then one can check that the special crossing $s_{i,j}$ occurs between strands in different cylinders SC_k and SC_r as defined above in (a). This implies that the braid β admits an exchange move. Therefore we have:

Proposition 5.1. *If K is a braided satellite knot written with the above notations as $K = \bar{\beta} \clubsuit \bar{\alpha}$. Then a reducible braid $\beta \clubsuit \alpha \in Bi(K)$, with $\beta \in Bi(C)$, admits an exchange move if and only if β admits an exchange move.*

This proposition implies that if a knot C has the property that no braid in $Bi(C)$ admits an exchange move then for any $\alpha \in B_n$ such that $\bar{\alpha}$ is a knot, the braided satellite knot $K = \bar{\beta} \clubsuit \bar{\alpha}$, $\beta \in Bi(C)$, has a braid index set with the same number of conjugacy class as $Bi(C)$. In particular if C is a torus knot then $Bi(K)$ has only one conjugacy class for any braid $\alpha \in B_n$.

We end this paper by an example of computation of the factor Λ, we prove:

Proposition 5.2. *The square knot S and the granny knot G have distinct factor Λ, namely:*

$$\Lambda_S = 10.9083.. \text{ and } \Lambda_G = 6.8541\dots$$

Let us recall (see [Ro]) that the granny knot G is the connected sum of two right-hand trefoil knots and the square knot S is the connected sum of a right-hand and a left-hand trefoil knots (see Fig. 4). The right-hand trefoil is the closure of $\sigma_1^3 \in B_2$ and the left-hand trefoil is the closure of $\sigma_1^{-3} \in B_2$. It is then easy to check that the granny knot is the closure of $\gamma = \sigma_1^3 \cdot \sigma_2^3 \in B_3$ and the square knot is the closure of $\Sigma = \sigma_1^3 \cdot \sigma_2^{-3} \in B_3$. The braid index theorem of [5] for composite knots implies that $bi(G) = bi(S) = 3$. The classification theorem for braid index 3 links (see [7]) implies that $Bi(G)$ and $Bi(S)$ are reduced to a single conjugacy class. Therefore the factor Λ_G (resp Λ_S) is just the dilatation factor λ_γ (resp λ_Σ) of the braid γ (resp Σ).

We can compute the dilatation factors by applying the algorithm of [17]. We do not perform here the complete computations but we give the final result and the necessary information to reproduce it (by hand) without doing all the tests. We refer the interested reader to [17] for the definitions and the details. Let us first remark that the braids γ and Σ are of minimal length in their conjugacy classes. We can then derive the other braids of the conjugacy classes having the same minimal length. In what follows we consider the braids

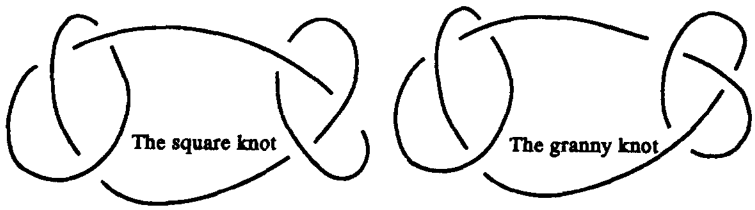

Fig. 4. The granny and the square knots.

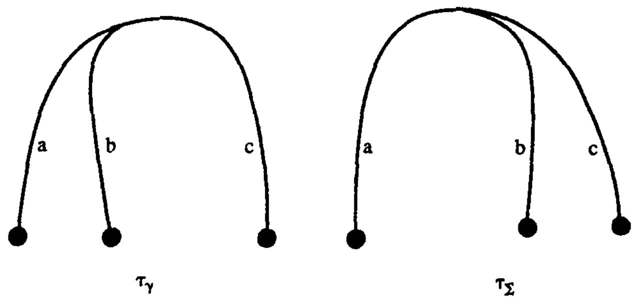

Fig. 5. The invariant train tracks.

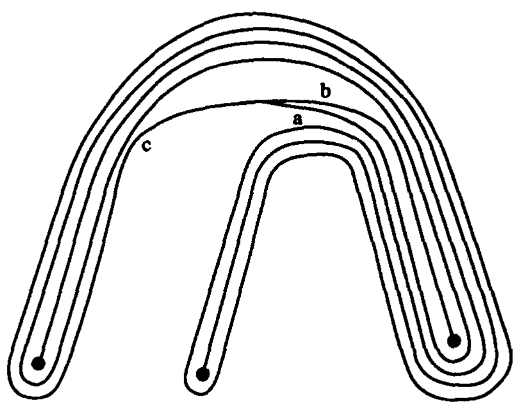

Fig. 6(a). The image $f_{\gamma'}(\tau_\gamma)$.

$\gamma' = \sigma_1^2 \cdot \sigma_2^3 \cdot \sigma_1$ and $\Sigma' = \sigma_1 \cdot \sigma_2^{-3} \cdot \sigma_1^2$ which are conjugate to γ and Σ, respectively. We derive a train track $\tau_{\gamma'}$ (resp $\tau_{\Sigma'}$) which is invariant under an element of the isotopy class $[f_{\gamma'}]$ (resp $[f_{\Sigma'}]$). The Fig. 5 shows the invariant train tracks $\tau_{\gamma'}$ and $\tau_{\Sigma'}$ and the Fig. 6 shows the minimal image of these train tracks under an element of $[f_{\gamma'}]$ and $[f_{\Sigma'}]$. It turns out that these isotopy classes are both pseudo-Anosov which is not a surprise from Theorem 1.1.

From these pictures we then compute the incidence matrices which are, respectively:

$$M_\gamma = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & 4 \end{pmatrix} \quad \text{and} \quad M_\Sigma = \begin{pmatrix} 6 & 1 & 10 \\ 4 & 0 & 5 \\ 3 & 0 & 4 \end{pmatrix}$$

for the induced maps on $\tau_{\gamma'}$ and $\tau_{\Sigma'}$. The dilatation factors being the largest eigenvalues of these matrices, we find: $\lambda_\Sigma = 10.9083\ldots$, and $\lambda_\gamma = 6.8541\ldots$ We can check furthermore that

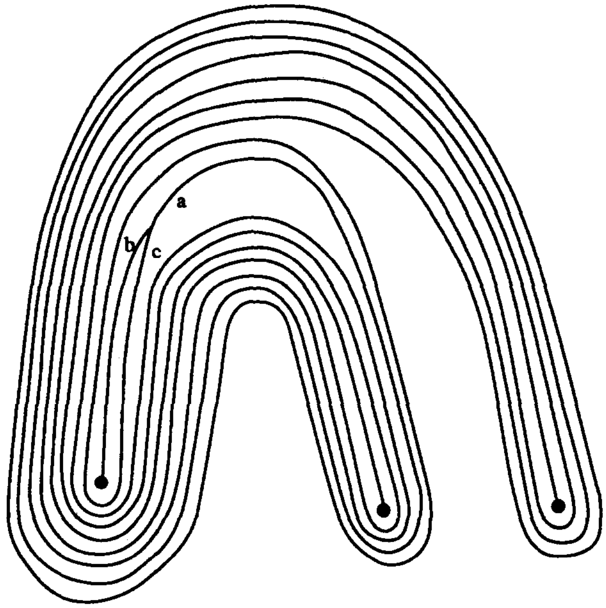

Fig. 6(b). The image $f_{\Sigma'}$ $(\tau_{\Sigma'})$.

the positive eigenvector (x_a, x_b, x_c) corresponding, for each of these matrix, to the largest eigenvalue satisfies one switch condition, namely: $x_a + x_b = x_c$ for M_γ and $x_a = x_b + x_c$ for M_Σ.

Acknowledgements—This paper was written during a visit at the University of California Santa Barbara under a CNRS grant. These institutions are acknowledged. The author is pleased to thank M. Boileau and K. Millett for their stimulating interest.

REFERENCES

1. E. Artin: Theorie der Zopfe. *Hamburg Abh*, **4** (1925), 47–72.
2. R. L. Adler, A. G. Konheim and M. H. McAndrew: Topological entropy, *Trans. A.M.S.* **114** (1965), 309–319.
3. M. Bestvina and M. Handel: Train tracks for surface homeomorphisms. Preprint (1991).
4. J. Birman: Braid, links and the mapping class group, *Ann. Math. Studies* **82** Princeton Univ. Press (1974).
5. J. Birman and W. Menasco: Studying links by closed braids IV: composite links and split links, *Invent. Math.* **102** (1990), 115–139.
6. J. Birman and W. Menasco: Special positions for essential tori in link complements. Preprint (1990) to appear in *Topology*.
7. J. Birman and W. Menasco: Studying links by closed braids III: classifying links which are closed 3-braids. To appear in *Pacific Journal of Math.*
8. R. Bowen: Entropy for group endomorphisms and homogeneous spaces, *Trans. A.M.S.* **153** (1970), 401–413.
9. L. E. J. Brouwer: Uber die periodischen transformationen der kugel, *Math. Ann.* **80** (1919), 39–41.
10. E. A. Elrifai and H. R. Morton: Algorithm for positive braids. Preprint (1990).
11. A. Fathi, F. Laudenbach and V. Poenaru: Travaux de Thurston sur les surfaces, *Astérisque*, **66/67**, (1979).
12. J. Franks and R. Williams: Braids and the Jones polynomial, *Trans. A.M.S.* **303** (1987), 97–108.
13. J. Hoste, A. Ocneanu, K. Millett, P. Freyd, W. Lickorish and D. Yetter: A new polynomial invariant of knots and links, *Bull. A.M.S.* **12** (1985), 257–267.
14. V. Jones: Heck algebra representation of braid groups and link polynomials, *Ann. Math.* **126** (1987), 335–388.
15. A. Katok: Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. Math. I.H.E.S.* **51** (1980), 137–174.
16. J. Los: Facteur de dilatation des entrelacs tressés. Preprint (1991).
17. J. Los: Pseudo-Anosov maps and invariant train track in the disc: a finite algorithm, *Proc. Lond. Math. Soc.* **66** (1993), 400–430.
18. J. Los: Topological entropy and periodic orbits for homeomorphisms of the disc: a lower bound. Preprint (1991).

19. H. R. MORTON: Closed braid representation for a link and its 2-variable polynomial. Preprint (1986).
20. H. R. MORTON: Infinitely many fibered knots having the same Alexander polynomial, *Topology* **17** (1978), 101–104.
21. J. NIELSEN: In *Collected papers of J. Nielsen.* Birkhauser (1986).
22. D. SILVER: Knot invariants from topological entropy. Preprint (1993).
23. W. P. THURSTON: On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. A.M.S.* **19** (1988), 415–431.
24. W. P. THURSTON: Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. A.M.S.* **6** (1982), 357–381.

Institut Non Linéaire de Nice
Université de Nice,
UMR 129, *CNRS*
1361 *Route des Lucioles*
06 560 *Valbonne*
France

and

University of California Santa Barbara
Mathematics Department
Santa Barbara, CA 93106
U.S.A